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Unconventional Geometric Quantum Computation in a Dissipative Cavity QED System Without the Heating

Chang-Yong Chen · Shuai Kang · Mang Feng

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Abstract We improve the scheme of geometric quantum phase gate (Chen et al. in Phys. Rev. A 74:032328, 2006) by using double-Hamiltonian evolution technique to remove the photon fluctuation in the cavity mode during the gating. We also shows that when the classical laser intensity is fixed, our gating time may be shorter than that in the ideal case due to the introduction of the cavity mode decay, although the dissipation decreases the corresponding fidelity and the success probability of the gate.

Keywords Geometric quantum phase gate · Fidelity · Success probability

Built-in fault-tolerant quantum gates have been achieved by means of the decoherence-free subspace [1, 2] and the geometric phase [3–12], which are different from quantum logic gates [13–19] by dynamical evolution being very sensitive to the parameter fluctuations in the operation. Geometric phase gates consist of the conventional geometric quantum gates (GQG) [3–11] and the unconventional ones [12, 20, 21]. We have noticed some quantum computing schemes based on the ideas of conventional GQGs by using super-conducting nanocircuits [6], NMR [8], semiconductor nanostructure [9], and trapped ions [10, 11]. These schemes are not focused on specific consideration about the influence from dissipation on the construction of GQGs. Recently, Pachos and Walther [20, 21] addressed specifically quantum computation with trapped $^{40}Ca^{+}$ ions in an optical cavity by employing adiabatic transitions and the quantum Zeno effect, considering the ionic spontaneous emission and the cavity decay. Cen and Zanardi [22] proposed double loop scheme to get rid of the negative influence of dissipation in the no-jump trajectory and to realize the geometric quantum computation. Fuentes-Guridi et al. [23] studied systematically non-Abelian adiabatic

C.-Y. Chen (✉) · S. Kang
Department of Physics and Information Engineering, Hunan Institute of Humanities, Science and Technology, Loudi 417000, People's Republic of China
e-mail: chenchangyong_64@hotmail.com

M. Feng
State Key Laboratory of Magnetic Resonance and Atomic and Molecular Physics, Wuhan Institute of Physics and Mathematics, Chinese Academy of Sciences, Wuhan 430071, People's Republic of China

holonomies for an open Markovian system and analyzed the robustness of holonomic quantum computation against decoherence. Carollo et al. [24, 25] have calculated the geometric phases related with the evolution of an open system subject to decoherence by a quantum-jump approach, giving some approximate results. Nevertheless, there has been actually no experiment achieved so far for the conventional GQGs. In contrast, for the unconventional GQGs, besides the theoretical proposals [12, 26], an experiment has been done with trapped ions [27]. Recently, a general displacement operator method for treatment of the unconventional GQGs has been proposed by Chen et al. [28]. However, the method can result in the heating effect since single-Hamiltonian evolution is applied.

In this paper, we use the general displacement operator method [28] and double-Hamiltonian evolution to investigate the geometric phase gate in a system with many identical three-level atoms confined in a cavity under decay, driven by a classical field. We propose an efficient scheme for implementation of the nonconventional GQG, based on a dissipative large-detuning interaction of the atoms with a cavity mode. We find that when the laser field intensity is fixed, the geometric quantum phase gating time is shorter than that in ideal case without the dissipation, by virtue of the technique of double-Hamiltonian evolution. The gating speed can be improved by increasing the laser field intensity. We also investigate analytically the influence of the cavity mode dissipation on the GQG, and show that the fidelity and the success probability of the GQG are lower in our scheme than in the ideal one. Our scheme can eliminate the heating effect as in [28]. Therefore, we argue that the nonconventional GQG can be implemented under the consideration of dissipation.

Consider N identical three-level atoms, each of which has one excited state $|i\rangle$ and two ground states $|e\rangle$ and $|g\rangle$. The qubits are encoded in the states $|e\rangle$ and $|g\rangle$, and the state $|i\rangle$ is an auxiliary state. The two levels $|e\rangle$ and $|i\rangle$ couple to the cavity mode with the coupling constant g and detuning $\Delta = \omega_0 - \omega_c$, assisted by a classical laser field with Rabi frequency Ω and detuning $\Delta - \delta = \omega_0 - \omega_L$, where $\delta \ll \Delta$ and $\delta = \omega_L - \omega_c$. As $|g\rangle$ is not involved in the interaction, under the consideration of the cavity decay and under the condition that no photon leaking out of the cavity is detected, the Hamiltonian [28] can be expressed as (assuming $\hbar = 1$)

$$\begin{aligned} H_s &= \omega_0 \sum_{j=1}^{N} S_{z,j} + \omega_c a^+ a + g \sum_{j=1}^{N} (a^+ S_j^- + a S_j^+) \\ &\quad + \Omega \sum_{j=1}^{N} (e^{-i(\omega_L t + \phi)} S_j^+ + e^{i(\omega_L t + \phi)} S_j^-) - i \frac{\kappa}{2} a^+ a, \end{aligned} \tag{1}$$

where ω_0, ω_c, and ω_L are the frequencies of the resonant transition between $|e\rangle$ and $|i\rangle$, the cavity mode, and the classical laser field, respectively, ϕ is the initial phase of the classical laser field, with $S_{z,j} = \frac{1}{2}(|i\rangle\langle i| - |e\rangle\langle e|)$, $S_j^+ = |i\rangle\langle e|$, and $S_j^- = |e\rangle\langle i|$, a^+ and a are the creation and annihilation operators of the cavity mode, respectively, and κ is the cavity decay rate. In the rotating frame with respect to the cavity frequency ω_c, $\Delta \gg \Omega, g$, and $\kappa t \ll \delta t$, the effective Hamiltonian [28–31] can be expressed as

$$\begin{aligned} H_{cei} = \sum_{j=1}^{N} \frac{\Omega g}{\Delta} \{ &[a e^{i(\delta + \frac{g^2}{\Delta})t} e^{-\kappa t/2} e^{i\phi} + a^+ e^{-i(\delta + \frac{g^2}{\Delta})t} e^{\kappa t/2} e^{-i\phi}] |i_j\rangle\langle i_j| \\ &- [a e^{i(\delta + \frac{g^2}{\Delta})t} e^{-\kappa t/2} e^{i\phi} + a^+ e^{-i(\delta + \frac{g^2}{\Delta})t} e^{\kappa t/2} e^{-i\phi}] |e_j\rangle\langle e_j| \} \\ &+ \frac{g^2}{\Delta} \sum_{i,j,i \neq j}^{N} (S_i^+ S_j^- + S_j^+ S_i^-), \end{aligned} \tag{2}$$

in which the only term that contributes an unconventional geometric phase shift to the evolution of the encoded qubit states $|g_j\rangle$ and $|e_j\rangle$ is

$$H'_{ceicg} = -\sum_{j=1}^{N} \frac{\Omega g}{\Delta}[a e^{i(\delta+\frac{g^2}{\Delta})t} e^{-\kappa t/2} e^{i\phi} + a^+ e^{-i(\delta+\frac{g^2}{\Delta})t} e^{\kappa t/2} e^{-i\phi}]|e_j\rangle\langle e_j|. \tag{3}$$

According to the definition of the general displacement operator [28], during the infinitesimal interval $[t, t+dt]$, the corresponding evolution in our qubit subspace is determined by

$$U(dt) = D[df_1(t), df_2(t)] = e^{-iH'_{ceicg}dt}, \tag{4}$$

where

$$df_1(t) = i\frac{\Omega g}{\Delta} e^{-i(\delta+\frac{g^2}{\Delta})t} e^{\kappa t/2} e^{i\phi} dt, \tag{5}$$

and

$$df_2(t) = -i\frac{\Omega g}{\Delta} e^{i(\delta+\frac{g^2}{\Delta})t} e^{-\kappa t/2} e^{-i\phi} dt. \tag{6}$$

Therefore, the qubit states including a single state $|e_j\rangle$ will evolve according to $D[df_1, df_2]$, and the one with two $|e_j\rangle$ will evolve according to $D[2df_1, 2df_2]$. Other states will remain unchanged. For simplicity, we assume from now on the cavity mode to be initially in a vacuum state $|0\rangle$. So after an interaction time t', the general displacement parameters $f_1$and f_2 can be, respectively expressed as

$$\begin{aligned} f_1(t') &= i\frac{\Omega g}{\Delta}\int_0^{t'} e^{-i(\delta+\frac{g^2}{\Delta})t} e^{\kappa t/2} e^{i\phi} dt \\ &= \frac{\Omega g}{\Delta}\frac{(\delta+g^2/\Delta) - i\kappa/2}{(\delta+g^2/\Delta)^2+\kappa^2/4}[1 - e^{-i(\delta+\frac{g^2}{\Delta})t'} e^{\kappa t'/2}] \times e^{i\phi}, \end{aligned} \tag{7}$$

and

$$\begin{aligned} f_2(t') &= -i\frac{\Omega g}{\Delta}\int_0^{t'} e^{i(\delta+\frac{g^2}{\Delta})t} e^{-\kappa t/2} dt \\ &= \frac{\Omega g}{\Delta}\frac{(\delta+g^2/\Delta) - i\kappa/2}{(\delta+g^2/\Delta)^2+\kappa^2/4}[1 - e^{i(\delta+\frac{g^2}{\Delta})t'} e^{-\kappa t'/2}] \times e^{-i\phi}. \end{aligned} \tag{8}$$

Then the geometric phase shifts $\Theta_{\kappa 1}$ and $\Theta'_{\kappa 1}$ [28], regarding $|e_j\rangle|g_k\rangle(|g_j\rangle|e_k\rangle)$ and $|e_j\rangle|e_k\rangle(j\neq k)$are

$$\begin{aligned} \Theta_{\kappa 1} &= \frac{1}{2}\mathrm{Im}\left[\int_\gamma f_2(t)df_1(t) - \int_\gamma f_1(t)df_2(t)\right] \\ &= \left(\frac{\Omega g}{\Delta}\right)^2 \frac{1}{[(\delta+g^2/\Delta)^2+\kappa^2/4]^2} \\ &\quad\times \left\{[(\delta+g^2/\Delta)^2-\kappa^2/4]\sin[(\delta+g^2/\Delta)t]\cosh\left(\frac{\kappa t}{2}\right)\right. \end{aligned}$$

$$+\kappa(\delta+g^2/\Delta)\cos[(\delta+g^2/\Delta)t]\sinh\left(\frac{\kappa t}{2}\right)$$
$$+[(\delta+g^2/\Delta)^2+\kappa^2/4](\delta+g^2/\Delta)t\Bigg\}, \tag{9}$$

and $\Theta'_{\kappa 1}=\frac{1}{2}\,\mathrm{Im}[\int_\gamma 2f_2(t)d2f_1(t)-\int_\gamma 2f_1(t)d2f_2(t)]=4\Theta_{\kappa 1}$, respectively. Moreover, we have terms $A_{\kappa 1}$ and $A'_{\kappa 1}$ related to the decay rates

$$\begin{aligned}A_{\kappa 1}&=-\frac{1}{2}\,\mathrm{Re}\left[\int_\gamma f_2(t)df_1(t)-\int_\gamma f_1(t)df_2(t)\right]\\&=\left(\frac{\Omega g}{\Delta}\right)^2\frac{1}{[(\delta+g^2/\Delta)^2+\kappa^2/4]^2}\\&\quad\times\Bigg\{[(\delta+g^2/\Delta)^2-\kappa^2/4]\cos[(\delta+g^2/\Delta)t]\sinh\left(\frac{\kappa t}{2}\right)\\&\quad-\kappa(\delta+g^2/\Delta)\sin[(\delta+g^2/\Delta)t]\cosh\left(\frac{\kappa t}{2}\right)\\&\quad+[(\delta+g^2/\Delta)^2+\kappa^2/4]\frac{\kappa t}{2}\Bigg\},\end{aligned} \tag{10}$$

and $A'_{\kappa 1}=-\frac{1}{2}\,\mathrm{Re}[\int_\gamma 2f_2(t)d2f_1(t)-\int_\gamma 2f_1(t)d2f_2(t)]=4A_{\kappa 1}$, respectively.

It is noted that the cavity decay κ results in impossibility of an exactly closed path movement of the general displacement in the parameter phase space when only single-Hamiltonian evolution is involved.

The second case is now considered that we change the phase of classical laser field as $\phi'=\phi+\pi$. Repeating the above derivation, we obtain the corresponding effective Hamiltonian that contributes an unconventional geometric phase shift to the evolution of the encoded qubit states $|g_j\rangle$ and $|e_j\rangle$

$$H''_{ceicg}=\sum_{j=1}^{N}\frac{\Omega g}{\Delta}[ae^{i(\delta+\frac{g^2}{\Delta})t}e^{-\kappa t/2}e^{i\phi}+a^+e^{-i(\delta+\frac{g^2}{\Delta})t}e^{\kappa t/2}e^{-i\phi}]|e_j\rangle\langle e_j|. \tag{11}$$

The corresponding evolution of the parameters are, respectively,

$$\begin{aligned}f'_1(t)&=-i\frac{\Omega g}{\Delta}\int_0^t e^{-i(\delta+\frac{g^2}{\Delta})t'}e^{\kappa t'/2}e^{i\phi}dt'\\&=-\frac{\Omega g}{\Delta}\frac{(\delta+g^2/\Delta)-i\kappa/2}{(\delta+g^2/\Delta)^2+\kappa^2/4}[1-e^{-i(\delta+\frac{g^2}{\Delta})t}e^{\kappa t/2}]\times e^{i\phi}\\&=-f_1(t),\end{aligned} \tag{12}$$

and

$$\begin{aligned}f'_2(t')&=i\frac{\Omega g}{\Delta}\int_0^t e^{i(\delta+\frac{g^2}{\Delta})t'}e^{-\kappa t'/2}dt\\&=-\frac{\Omega g}{\Delta}\frac{(\delta+g^2/\Delta)-i\kappa/2}{(\delta+g^2/\Delta)^2+\kappa^2/4}[1-e^{i(\delta+\frac{g^2}{\Delta})t}e^{-\kappa t/2}]\times e^{-i\phi}\end{aligned}$$

$$= -f_2(t). \tag{13}$$

Then the geometric phase shifts $\Theta_{\kappa 2}$ and $\Theta'_{\kappa 2}$ are, respectively,

$$\begin{aligned}\Theta_{\kappa 2} &= \frac{1}{2}\,\mathrm{Im}\left[\int_\gamma f_2'(t)df_1'(t) - \int_\gamma f_1'(t)df_2'(t)\right]\\ &= \left(\frac{\Omega g}{\Delta}\right)^2 \frac{1}{[(\delta+g^2/\Delta)^2+\kappa^2/4]^2}\\ &\quad\times\left\{[(\delta+g^2/\Delta)^2-\kappa^2/4]\sin[(\delta+g^2/\Delta)t]\cosh\left(\frac{\kappa t}{2}\right)\right.\\ &\quad+\kappa(\delta+g^2/\Delta)\cos[(\delta+g^2/\Delta)t]\sinh\left(\frac{\kappa t}{2}\right)\\ &\quad\left.+[(\delta+g^2/\Delta)^2+\kappa^2/4](\delta+g^2/\Delta)t\right\}\\ &= \Theta_{\kappa 1},\end{aligned} \tag{14}$$

and

$$\Theta'_{\kappa 2} = \frac{1}{2}\,\mathrm{Im}\left[\int_\gamma 2f_2'(t)d2f_1'(t) - \int_\gamma 2f_1'(t)d2f_2'(t)\right] = 4\Theta_{\kappa 2}. \tag{15}$$

Moreover, the amplitude damping terms $A_{\kappa 2}$ and $A'_{\kappa 2}$ are given by

$$\begin{aligned}A_{\kappa 2} &= -\frac{1}{2}\,\mathrm{Re}\left[\int_\gamma f_2'(t)df_1'(t) - \int_\gamma f_1'(t)df_2'(t)\right]\\ &= \left(\frac{\Omega g}{\Delta}\right)^2 \frac{1}{[(\delta+g^2/\Delta)^2+\kappa^2/4]^2}\\ &\quad\times\left\{[(\delta+g^2/\Delta)^2-\kappa^2/4]\cos[(\delta+g^2/\Delta)t]\sinh\left(\frac{\kappa t}{2}\right)\right.\\ &\quad-\kappa(\delta+g^2/\Delta)\sin[(\delta+g^2/\Delta)t]\cosh\left(\frac{\kappa t}{2}\right)\\ &\quad\left.+[(\delta+g^2/\Delta)^2+\kappa^2/4]\frac{\kappa t}{2}\right\}\\ &= A_{\kappa 1},\end{aligned} \tag{16}$$

and

$$A'_{\kappa 2} = -\frac{1}{2}\,\mathrm{Re}\left[\int_\gamma 2f_2'(t)d2f_1'(t) - \int_\gamma 2f_1'(t)d2f_2'(t)\right] = 4A_{\kappa 2} = 4A_{\kappa 1}. \tag{17}$$

Next, we consider the two-qubit unconventional geometric phase gate. Assume that $N = 2$ and the atoms are first illuminated with a classical laser with phase ϕ for a time $\tau/2$. Then the phase is shifted by π and the atoms are illuminated for another time $\tau/2$. If the cavity is initially in the vacuum state $|0\rangle$, we give below the evolution of the system with similar steps to [26, 28, 29],

$$
\begin{aligned}
|g\rangle_1|g_2\rangle|0\rangle &\to |g\rangle_1|g_2\rangle|0\rangle \to |g\rangle_1|g_2\rangle|0\rangle,\\
|g\rangle_1|e\rangle_2|0\rangle &\to e^{-i\Omega^2\tau/\Delta}e^{-A_{\kappa 1}(\tau)}e^{i\Theta_{\kappa 1}(\tau)}D[0,0]|g\rangle_1|e\rangle_2|0\rangle\\
&\quad \to e^{-A_{\kappa 1}(\tau)}e^{i[\Theta_{\kappa 1}(\tau)-\Omega^{2\tau}/\Delta)]}|g\rangle_1|e\rangle_2|0\rangle,\\
|e\rangle_1|g\rangle_2|0\rangle &\to e^{-i\Omega^2\tau/\Delta}e^{-A_{\kappa 1}(\tau)}e^{i\Theta_{\kappa 1}(\tau)}D[0,0]|e\rangle_1|g\rangle_2|0\rangle\\
&\quad \to e^{-A_{\kappa 1}(\tau)}e^{i[\Theta_{\kappa 1}(\tau)-\Omega^2\tau/\Delta)]}|e\rangle_1|g\rangle_2|0\rangle,\\
|e\rangle_1|e\rangle_2|0\rangle &\to e^{-i2\Omega^2\tau/\Delta}e^{-A'_{\kappa 1}(\tau)}e^{i\Theta'_{\kappa 1}(\tau)}D[0,0]|e\rangle_1|e\rangle_2|0\rangle\\
&\quad \to e^{-4A_{\kappa 1}(\tau)}e^{i[4\Theta_{\kappa 1}(\tau)-2\Omega^2\tau/\Delta)]}|e\rangle_1|e\rangle_2|0\rangle,
\end{aligned}
\tag{18}
$$

where

$$
\begin{aligned}
\Theta_{\kappa 1}(\tau) = \left(\frac{\Omega g}{\Delta}\right)^2 & \frac{1}{[(\delta+g^2/\Delta)^2+\kappa^2/4]^2}\\
&\times\Bigg\{[(\delta+g^2/\Delta)^2-\kappa^2/4]\sin[(\delta+g^2/\Delta)\tau]\cosh\left(\frac{\kappa\tau}{2}\right)\\
&+\kappa(\delta+g^2/\Delta)\cos[(\delta+g^2/\Delta)\tau]\sinh\left(\frac{\kappa\tau}{2}\right)\\
&+[(\delta+g^2/\Delta)^2+\kappa^2/4](\delta+g^2/\Delta)\tau\Bigg\},
\end{aligned}
\tag{19}
$$

and

$$
\begin{aligned}
A_{\kappa 1}(\tau) = \left(\frac{\Omega g}{\Delta}\right)^2 & \frac{1}{[(\delta+g^2/\Delta)^2+\kappa^2/4]^2}\\
&\times\Bigg\{[(\delta+g^2/\Delta)^2-\kappa^2/4]\cos[(\delta+g^2/\Delta)\tau]\sinh\left(\frac{\kappa\tau}{2}\right)\\
&-\kappa(\delta+g^2/\Delta)\sin[(\delta+g^2/\Delta)\tau]\cosh\left(\frac{\kappa\tau}{2}\right)\\
&+[(\delta+g^2/\Delta)^2+\kappa^2/4]\frac{\kappa\tau}{2}\Bigg\},
\end{aligned}
\tag{20}
$$

The displacement of the cavity mode in the first time $\tau/2$ is concealed by that in the second interaction time $\tau/2$, and the displacement is along a closed path during the time τ. Therefore, the mode returns to the original point in phase space, with the system obtaining a conditional geometric phase, depending on the sum of the areas enclosed by the displacement loop of the cavity mode in the phase space. In the computational qubit subspace $\{|g\rangle_j, |e\rangle_j, j=1,2\}$, with the single qubit operation $|e\rangle_j \to e^{-i[\Theta_{\kappa 1}(\tau)-\Omega^2\tau/\Delta]}|e\rangle_j$ $(j=1,2)$, the $2\Theta_{\kappa 1}$-geometric phase gate can be obtained

$$|g\rangle_1|g\rangle_2|0\rangle \to |g\rangle_1|g\rangle_2|0\rangle,$$

$$
\begin{aligned}
&|g\rangle_1|e\rangle_2|0\rangle \to e^{-A_{\kappa 1}}|g\rangle_1|e\rangle_2|0\rangle,\\
&|e\rangle_1|g\rangle_2|0\rangle \to e^{-A_{\kappa 1}}|e\rangle_1|g\rangle_2|0\rangle,\\
&|e\rangle_1|e\rangle_2|0\rangle \to e^{-4A_{\kappa 1}}e^{i2\Theta_{\kappa 1}(\tau)}|e\rangle_1|e\rangle_2|0\rangle,
\end{aligned}
\tag{21}
$$

where $2\Theta_{\kappa 1}(\tau) = \pi$ yields an approximate π-phase gate for the two atoms (qubits) under the influence from the cavity decay, Different from the ideal case in [26], where the phase gate does not involve the damping factors, and the dissipation case in [28], in which the coherent states $|f_1\rangle$ and $|2f_1\rangle$ are involved in the geometric gate, (21) includes only the amplitude damping factors, no coherent states in the cavity mode after the gate operation. Therefore, our scheme is not sensitive to the heating and there is no the additional phase shift originated from the general displacement operator in this scheme. The required condition for this gating is

$$
\begin{aligned}
&\left(\frac{\Omega g}{\Delta}\right)^2 \frac{1}{[(\delta+g^2/\Delta)^2+\kappa^2/4]^2}\\
&\quad\times\left\{[(\delta+g^2/\Delta)^2-\kappa^2/4]\sin[(\delta+g^2/\Delta)\tau]\cosh\left(\frac{\kappa\tau}{2}\right)\right.\\
&\quad+\kappa(\delta+g^2/\Delta)\cos[(\delta+g^2/\Delta)\tau]\sinh\left(\frac{\kappa\tau}{2}\right)\\
&\quad\left.+[(\delta+g^2/\Delta)^2+\kappa^2/4](\delta+g^2/\Delta)\tau\right\}\\
&=\frac{\pi}{2}.
\end{aligned}
\tag{22}
$$

If assume that $(\delta + g^2/\Delta)\tau = 2\pi$, when $\kappa\pi/(\delta + g^2/\Delta) \ll 1$, we have

$$
\Omega \simeq \frac{\Delta}{2g}\times(\delta+g^2/\Delta)\times\left[1-\frac{\kappa^2}{(\delta+g^2/\Delta)^2}\right],
\tag{23}
$$

from which we know that when the classical laser intensity is fixed, $(\delta + g^2/\Delta)$ will become increasing due to the introduction of the cavity mode decay κ, leading to a shorter gating time than that in the ideal case. Apparently, the gating speed can be further improved by increasing the classical laser intensity. Therefore, the high-speed geometric phase gating can be implemented. Naturally, the corresponding amplitude damping factors are

$$
\begin{aligned}
A_{\kappa 1}(\tau) &= \frac{\pi}{2}\left\{[(\delta+g^2/\Delta)^2-\kappa^2/4]\sinh\left[\frac{\kappa\pi}{(\delta+g^2/\Delta)}\right]\right.\\
&\quad\left.+[(\delta+g^2/\Delta)^2+\kappa^2/4]\frac{\kappa\pi}{(\delta+g^2/\Delta)}\right\}\\
&\quad\times\left\{\kappa(\delta+g^2/\Delta)\sinh\left[\frac{\kappa\pi}{(\delta+g^2/\Delta)}\right]+2[(\delta+g^2/\Delta)^2+\kappa^2/4]\pi\right\}^{-1}\\
&\simeq\frac{\kappa\pi}{(\delta+g^2/\Delta)}\times\left\{1-\frac{2\kappa^2\pi^2}{3(\delta+g^2/\Delta)^2}\right\},
\end{aligned}
\tag{24}
$$

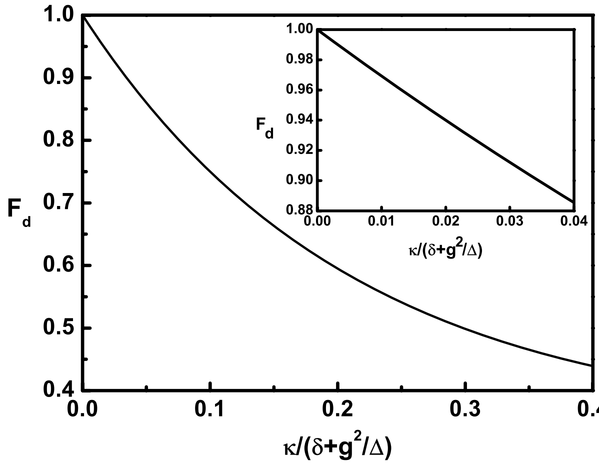

Fig. 1 The fedility F_d between the state after the ideal controlled-NOT gate and that after the approximate controlled-NOT gate, vs. $(\kappa/(\delta + g^2/\Delta))$ based on (27)

$$A_{\kappa 2}(\tau) = 4A_{\kappa 1}(\tau) \simeq \frac{4\kappa\pi}{(\delta + g^2/\Delta)} \times \left\{1 - \frac{2\kappa^2\pi^2}{3(\delta + g^2/\Delta)^2}\right\}. \tag{25}$$

It is revealed that the amplitude damping factors will increase with the cavity mode decay rate κ increasing, resulting in lowering of the fidelity and success probability of the geometric phase gate.

From (21), an approximate two-qubit controlled-NOT gate in the computational subspace $\{|g\rangle_1|g\rangle_2, |g\rangle_1|e\rangle_2, |e\rangle_1|g\rangle_2, |e\rangle_1|e\rangle_2\}$ can be constructed as

$$C_{apr\text{-}not} = H_2^+ P H_2 = \frac{1}{2}\begin{bmatrix} 1+e^{-A_{\kappa 1}(\tau)} & 1-e^{-A_{\kappa 1}(\tau)} & 0 & 0 \\ 1-e^{-A_{\kappa 1}(\tau)} & 1+e^{-A_{\kappa 1}(\tau)} & 0 & 0 \\ 0 & 0 & e^{-A_{\kappa 1}(\tau)} - e^{-4A_{\kappa 1}(\tau)} & e^{-A_{\kappa 1}(\tau)} + e^{-4A_{\kappa 1}(\tau)} \\ 0 & 0 & e^{-A_{\kappa 1}(\tau)} + e^{-4A_{\kappa 1}(\tau)} & e^{-A_{\kappa 1}(\tau)} - e^{-4A_{\kappa 1}(\tau)} \end{bmatrix}, \tag{26}$$

where H_2 and P represent the Hadamard gate on the second qubit and the geometric phase gate in (21).

Following it, we have in detail studied the error by applications of the controlled-NOT gate, i.e., (26) on two qubits (atoms) initially prepared in the product state $|\psi_0\rangle = (|g\rangle_1 + |e\rangle_1)(|g\rangle_2 + |e\rangle_2)/2$. It is noted that the state can be preserved unchanged under action of the ideal controlled-NOT gate C_{idnot}, that is, $C_{idnot}|\psi_0\rangle = |\psi_0\rangle$. Since the definition of the fidelity is the overlap between the state after the ideal gate operation and that after the real operations with dissipation, we have the fidelity F_d, which described the relation between the state after the ideal controlled-NOT gate and that after the approximate controlled-NOT gate, vs. $(\kappa/(\Delta + g^2/\Delta))$ in Fig. 1, after the $C_{apr\text{-}not}$ gating [20, 21, 28],

$$F_d = |\langle\psi_0|C_{apr\text{-}not}|\psi_0\rangle|^2 = \frac{1}{4}[1 + e^{-A_{\kappa 1}(\tau)}]^2. \tag{27}$$

In fact, we can also give the fidelity and success probability after each geometric phase gate based on (21), which are $F_{p-d} = \frac{1}{4}\cdot\frac{(1+2e^{-A_{\kappa 1}}+e^{-4A_{\kappa 1}})^2}{(1+2e^{-2A_{\kappa 1}}+e^{-8A_{\kappa 1}})}$ and $P_{p-d} = \frac{1}{4}(1 + 2e^{-2A_{\kappa 1}} + e^{-8A_{\kappa 1}})$, respectively, when a phase gate operation on the initial state $|\psi_0\rangle = (|g\rangle_1 + |e\rangle_1)(|g\rangle_2 + |e\rangle_2)/2$ is completed. Since we can analytically describe the influence from the cavity decay, we could demonstrate the validity of an approximate geometric phase gate by means of (18), (21), and (22). Some detailed discussion related to the fidelity and the success probability are analogue to those in [28].

Here we briefly discuss the experimental feasibility of our scheme in the present microwave regime, in which the lifetime of the Rydberg atomic excited state is about $T_r = 3 \times 10^{-2}$ s, the coupling strength between the cavity mode and the atoms is $g = 2\pi \times 49$ kHz, and the photon lifetime in the cavity is $T_c = 3.0 \times 10^{-3}$ s [17, 19]. Direct algebra from (22) yields that, our gating time is $\tau \simeq 10^{-4}$ s, much shorter than T_c, and T_r. Therefore, our scheme can be implemented by current cavity QED technology. Moreover, our treatment involving the cavity decay has shown the possibility of high fidelity and high success rate of our scheme.

In conclusion, we have utilized the general displacement operator method and the double-Hamiltonian evolution technique [32, 33] to investigate the geometric quantum computation under the consideration of the cavity mode dissipation. We find that the geometric phase gate can be implemented in the case of dissipation, the gating time of which is a little shorter than that in the ideal condition. The strengths of the displacement drives do not need to be much smaller than the cavity frequency, allowing the generation of a high-speed geometric gate. Therefore, the heating of the cavity mode is negligible. Different from [28], where the completely closed path in an open phase space is unavailable, it can be formed in our scheme where the heating effect and the additional phase shift have been eliminated due to the introduction of the double-Hamiltonian evolution [32, 33]. Our concrete study of the two-atom case has shown that our scheme works with a little shorter gating time (due to the cavity decay) than in a previous proposal [26], as long as the implementation time is much shorter than the cavity decay time. Therefore, a non-ideal GQG is available after this minor modification. Different from the methods in [22–25], our approach presents analytical expressions for the influence from the cavity decay on a geometric quantum computation. As the method employed in the present paper has more advantages than in [28], it is applicable to the case involving atomic spontaneous emission, which will be given elsewhere.

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